Inquisitive semantics

—NASSLLI 2012 lecture notes—

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June 17, 2012

About this document
These are lecture notes for a course on inquisitive semantics at NASSLLI 2012. The current document contains the first six sections of the lecture notes. Two further sections will be added during the course. Comments of any kind are of course more than welcome. The most recent version of the notes will be posted on the course website:

https://sites.google.com/site/inquisitivesemantics/courses/nasslli-2012

Contents

1 Introduction

The central aim of inquisitive semantics is to develop a notion of semantic meaning that captures both *informative* and *inquisitive* content. This enriched notion of meaning is intended to provide a new foundation for the analysis of linguistic discourse, in particular the type of discourse that is aimed at exchanging information.

The classical truth-conditional notion of meaning embodies the informative content of sentences, and thereby reflects the descriptive use of language. Stalnaker (1978) gave this notion a dynamic and conversational twist by taking the meaning of a sentence to be its potential to change the common ground, i.e., the body of shared information established in a conversation. The notion of meaning that resulted from this 'dynamic turn' reflects the active use of language in changing information. However, it does not yet reflect the *inter*active use of language in *ex*changing information. This requires yet another turn, an 'inquisitive turn', leading to a notion of meaning that directly reflects the nature of information exchange as a cooperative process of raising and resolving issues.

These lecture notes bring together and further expand on a number of results obtained in our recent work on inquisitive semantics Groenendijk and Roelofsen (2009); Ciardelli (2009); Ciardelli and Roelofsen (2011); Roelofsen (2011); Ciardelli *et al.* (2012).[1] In particular, they provide a detailed exposition of what we currently see as the most basic implementation of the framework, which we refer to as $\mathsf{Inq}_{\mathrm{B}}$.

The notes are organized as follows. Section 2 introduces the new notion of meaning that forms the heart of inquisitive semantics. Section 3 identifies the algebraic structure of this new space of meanings, and section 4 presents a system that associates meanings with sentences in a first-order language. The basic logical properties of this system are characterized in section 5, and finally, its relevance for natural language semantics is discussed in section 6.

2 Inquisitive meanings

A general scheme in which many notions of meaning can be naturally framed is the following. When a sentence is uttered in a certain *discourse context*, it

[1]For more recent and ongoing work on inquisitive semantics we refer to www.illc.uva.nl/inquisitive-semantics.

expresses a *proposition*, which embodies a proposal to change the context in certain ways. This proposition is determined by the *meaning* of the sentence. The meaning of a sentence, therefore, can be thought of as something which embodies the potential of a sentence to express a proposition when uttered in a discourse context: formally, it is a function from contexts to propositions.

This general scheme gives rise to different notions of meaning depending on how the two parameters occurring in it, namely the notions of *discourse context* and *proposition*, are instantiated. In this section we will work our way to the inquisitive notion of meaning by motivating and illustrating the notion of context and proposition which are adopted in inquisitive semantics, as well as the particular restrictions imposed on meaning functions.

2.1 Information states and issues

In this section we introduce the two basic formal notions that play a role in the inquisitive picture of meaning: *information states* and *issues*.

Information states. We adopt the standard notion of an information state as a set of possible worlds. Throughout the discussion we assume a fixed set ω of possible worlds, whose nature will depend on the choice of a formal language.

Definition 1 (Information states)**.**
An *information state* is a set $s \subseteq \omega$ of possible worlds.

We will often refer to an information state simply as a *state*. A state s is thought of as representing the information that the actual world lies in s. If $t \subseteq s$, then t locates the actual world at least as precisely as s: we may thus call t an *enhancement* of s.

Definition 2 (Enhancements)**.**
A state t is called an *enhancement* of s just in case $t \subseteq s$.

Notice that this definition includes the *trivial* enhancement $t = s$. If an enhancement t is non-trivial, that is, if t is *strictly* more informed than s, then we call t a *proper* enhancement of s.

The two extreme cases of the enhancement order are the empty state $\emptyset$, which is an enhancement of any state, and the set ω of all worlds, of which any state is an enhancement. The former models a state in which any

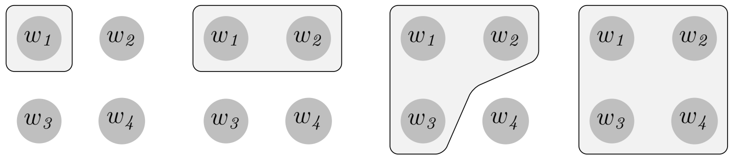


Figure 1: Information states.

possible world has been discarded as a candidate for the actual world, that is, the available information is inconsistent: we call $\emptyset$ the *inconsistent state*. The latter, on the contrary, models a state in which any possible world is taken to be a plausible candidate for the actual world, that is, we have no clue at all what the actual world is like: we call ω the *ignorant state*.

Figure 1 shows some examples of information states when our set of possible worlds consists of four worlds: w_1, w_2, w_3, w_4. Notice that these states are arranged according to the enhancement ordering from left to right, the last figure depicting the ignorant state ω.

Issues. An issue is meant to represent the semantic content of a request for information. Now, what does it mean to request information? If the information available in a certain context is represented by a state s, a request for information in s is a request to locate the actual world more precisely inside s. Thus, the content of the request consists in a specification of which enhancements of s locate the actual world with sufficient precision. Hence, an issue in s will be modeled as a non-empty set I of enhancements of s.

Importantly, we do not regard just *any* non-empty set I of enhancements of s as an issue. First, if I contains a certain enhancement t of s, and $t' \subseteq t$ is a further enhancement of t, then t' must also be in I. After all, if t locates the actual world with sufficient precision, then t' cannot fail to do so as well. So, I must be *downward closed*.

Second, the elements of I must together form a *cover* of s. That is, every world in s must be included in at least one element of I. To see why this is a natural requirement, suppose that w is a world in s that is not included in any element of I. Then the information available in s does not preclude w from being the actual world. But if w is indeed the actual world, then it would be impossible to satisfy the request represented by I without discarding the actual world. Thus, in order to ensure that it is possible to satisfy the request

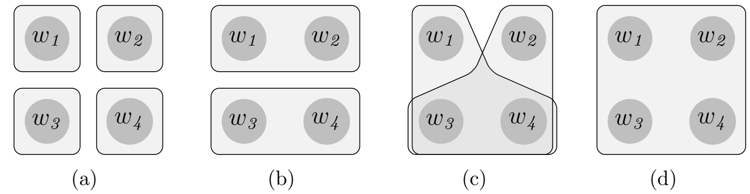


Figure 2: Issues over the state $\{w1, w2, w3, w4\}$.

represented by I without discarding the actual world, I should form a cover of s. This leads us to the following notion of an issue.

Definition 3 (Issues)**.**
Let s be an information state, and I a non-empty set of enhancements of s. Then we say that I is an issue over s if and only if:

1. I is *downward closed*: if $t \in I$ and $t' \subseteq t$ then also $t' \in I$
2. I forms a *cover* of s: $\bigcup I = s$

Definition 4 (Settling an issue)**.**
Let s be an information state, t an enhancement of s, and I an issue over s. Then we say that t settles I if and only if $t \in I$.

Notice that an issue I over a state s may contain s itself. This means that I does not request any information beyond the information that is already available in s. We call I a *trivial* issue over s in this case. Downward closure implies that for any state s there is precisely one trivial issue over s, namely $\wp(s)$. On the other hand, if $s \notin I$, then in order to settle I further information is required, that is, a proper enhancement of s must be established. In this case we call I a *proper* issue.

Two issues over a state s can be compared in terms of the information that they request: one issue I is at least as inquisitive as another issue J in case any state that settles I also settles J. Since an issue is identified with the set of states that settle it, we obtain the following definition.

Definition 5 (Ordering issues)**.**
Given two issues I, J on a state s, we say that I is *at least as inquisitive* as J just in case $I \subseteq J$.

Among the issues over a state s there is always a *least* and a *most* inquisitive one. The least inquisitive issue over s is the trivial issue $\wp(s)$ which, as we saw, requests *no* information. The most inquisitive issue over s is $\{\{w\} \mid w \in s\} \cup \{\emptyset\}$, which can only be settled consistently by providing a complete description of what the actual world is like.

Figure 2 shows some issues over the information state $s = \{w_1, w_2, w_3, w_4\}$. In order to keep the figures neat, only the maximal elements of these issues are displayed. The issue depicted in (a) is the most inquisitive issue over s, which is only settled by specifying precisely which world is the actual one. The issue depicted in (b) is settled either by locating the actual world within the set $\{w_1, w_2\}$, or by locating it within $\{w_3, w_4\}$. The issue depicted in (c) is settled either by locating the actual world in $\{w_1, w_3, w_4\}$, or by locating it within $\{w_2, w_3, w_4\}$. Finally, (d) represents the trivial issue over s, which is *already* settled in s. Both (b) and (c) are less inquisitive than (a) and more inquisitive than (d), while they are incomparable with each other.

2.2 Discourse contexts and propositions

The notions of information states and issues introduced in the previous section constitute the basic ingredients of the inquisitive picture of meaning. In this section they will be put to use to define the notions of *discourse contexts* and *propositions*.

Discourse contexts. For our present purposes, a discourse context can be identified with the *common ground* of the discourse, that is, the body of information that is common knowledge among the conversational participants. A discourse context will thus be modeled simply as an information state.

Propositions. The proposition expressed by a sentence in a certain discourse context should embody the effect of an utterance of that sentence in that discourse context. We recognize two types of effects that an utterance may have. First, as is assumed in traditional accounts of meaning, a sentence can be used to *provide* information, that is, to enhance the information state of the common ground. Thus, a proposition may specify an enhancement t of the current context s.

However, unlike those traditional accounts, we take sentences to have the potential to do *more* than just providing information. Namely, sentences may

also be used to *request* information. Therefore, besides specifying a certain enhancement t of the current context s, a proposition may also specify a certain *issue* I.

Now, we might take the stance that there are two distinct types of propositions: *informative* propositions, which specify an enhancement of the current context, and *inquisitive* propositions, which specify an issue over the current context. However, we will take a more general and unifying standpoint, which assumes only one, richer type of proposition. Namely, we will define a proposition A over a context s as having both an informative component, consisting in an enhancement t of s, and an inquisitive component, consisting in an issue I over t.

Definition 6 (Propositions, to be simplified presently)**.**
A *proposition* over a state s is a pair $A = (t, I)$, where:

- t is an enhancement of s called the *informative content* of A;
- I is an issue over t called the *inquisitive content* of A.[2]

Consider a proposition $A = (t, I)$. Since I is an issue over t, the union of all the states in I must coincide with t. This means that the informative component t of the proposition can be retrieved from I, and it need not appear explicitly in the representation of the proposition. Thus, our notion of propositions can be simplified as follows.

Definition 7 (Propositions, simplified)**.**

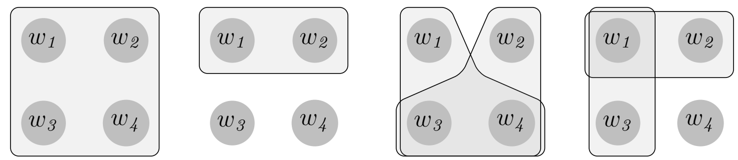
- A *proposition* A over a state s is an issue over a substate $t \subseteq s$.
- We denote by Π_s the set of all propositions over a state s.

Notice that if A is a proposition over s, it is also a proposition over any state $t \supseteq s$. In particular, any proposition is a proposition over the ignorant state ω: therefore, we simply write Π for Π_ω, and we call Π the *set of propositions*.

The informative content of a proposition A is embodied by $\bigcup A$. We will denote this set of worlds as $\mathsf{info}(A)$.

Definition 8 (Informative content)**.** For any $A \in \Pi$: $\mathsf{info}(A) := \bigcup A$

[2] This is incompatible with what happens later in the semantics section.



(a) −informative −inquisitive (b) +informative −inquisitive (c) −informative +inquisitive (d) +informative +inquisitive

Figure 3: Propositions over the state $\{w_1, w_2, w_3, w_4\}$.

In expressing a proposition A, one provides the information that the actual world lies in $\mathsf{info}(A)$ and requests enough information from other participants to locate the actual world inside one of the states in A.

The fact that we take any proposition to have both an informative and an inquisitive component should not be taken to mean that we take any proposition to be both informative and inquisitive: for either, or even both, of these components may be trivial. We say that a proposition is *informative* only in case its informative content is non-trivial, and *inquisitive* only in case its inquisitive content is non-trivial.

Definition 9 (Informativeness, inquisitiveness)**.**
Let A be a proposition over a state s.

- We say that A is *informative* in s just in case $\mathsf{info}(A) \subset s$;
- We say that A is *inquisitive* in s just in case $\mathsf{info}(A) \notin A$.

So, we still have *purely informative* propositions, which do not request any information, and *purely inquisitive* propositions, which do not provide any information. However, both are regarded as particular cases of a unique notion of proposition. Moreover, we are not committed to the assumption that every proposition is either purely informative or purely inquisitive: our approach leaves room for *hybrid* propositions, which are both informative and inquisitive at the same time.

In figure 3, several examples of propositions over $s = \{w_1, w_2, w_3, w_4\}$ are depicted, always with the convention that only the maximal elements are displayed. The proposition represented in figure (a) is not informative in s, because its informative content coincides with s, and it is also not inquisitive in s, because its inquisitive content forms a trivial issue over s.

The proposition represented in figure (b) is informative in s, because its informative content is a proper enhancement of s, but not inquisitive in s, since its inquisitive content is trivial. The proposition represented in figure (c) is not informative in s, because its informative content coincides with s, but it is inquisitive, because its inquisitive content is non-trivial. Finally, the proposition represented in figure (d) is both informative and inquisitive, since its informative content is a proper enhancement of s and its inquisitive content is non-trivial.

Propositions can be compared in terms of the information that they provide and the information that they request. We say that a proposition A is *at least as informative* as a proposition B if the informative content of A is an enhancement of the informative content of B.

Definition 10 (Informativeness ordering)**.** Let $A, B \in \Pi$.
We say that A is *at least as informative* as B just in case $\mathsf{info}(A) \subseteq \mathsf{info}(B)$.

If A and B are equally informative propositions, then we say that A is at least as inquisitive as B in case the inquisitive content of A is at least as demanding as the inquisitive content of B.

Definition 11 (Inquisitiveness ordering)**.** Let $A, B \in \Pi$, $\mathsf{info}(A) = \mathsf{info}(B)$.
Then we say that A is *at least as inquisitive* as B just in case $A \subseteq B$.

Now we would like to say that a proposition A *entails* a proposition B just in case A is both at least as informative and at least as inquisitive as B. But there is a subtlety here. Namely, if A is strictly more informative than B, then A and B cannot be compared directly in terms of inquisitiveness. Thus, what we request is that A be at least as informative as B and moreover, that A be at least as inquisitive as *the restriction of B to* $\mathsf{info}(A)$.

Definition 12 (Restriction)**.**
If $A \in \Pi_s$ and $t \subseteq s$, the *restriction* of A to t is the proposition $A{\upharpoonright}t \in \Pi_t$ defined by:

$$A{\upharpoonright}t = \{t' \subseteq t \mid t' \in A\}$$

Intuitively, $A{\upharpoonright}t$ is a proposition over t that inherits the content of A. The informative content of $A{\upharpoonright}t$ amounts to $\mathsf{info}(A) \cap t$. Thus, $A{\upharpoonright}t$ provides the information that the actual world lies in $\mathsf{info}(A)$, which is precisely the information provided by A itself. Moreover, the request expressed by $A{\upharpoonright}t$ is to enhance $\mathsf{info}(A\restriction t)$ in such a way as to satisfy the issue A. Notice that if A itself is already a proposition over t, then $A\restriction t$ simply amounts to A.

The notion of restriction allows us to define entailment between propositions. A proposition A *entails* a proposition B in case (i) A is at least as informative as B: $\mathsf{info}(A) \subseteq \mathsf{info}(B)$; and (ii) A is at least as inquisitive as the restriction of B to $\mathsf{info}(A)$: $A \subseteq B\restriction\mathsf{info}(A)$. However, it is easy to see that these two conditions are satisfied if and only if $A \subseteq B$. Therefore, entailment between propositions can simply be defined as inclusion.

Definition 13 (Entailment between propositions)**.**
Let $A, B \in \Pi$. Then we say that A *entails* B just in case $A \subseteq B$.

2.3 Meanings

Equipped with formal notions of discourse contexts and propositions, let us now come back to our initial picture of meaning. In every discourse context, a sentence expresses a certain proposition. This proposition is determined by the meaning of the sentence. The meaning of a sentence, therefore, consists in the potential that the sentence has to express propositions in context. It can be modeled as a function f that maps each context s to a proposition $f(s) \in \Pi_s$. We call $f(s)$ the proposition *expressed* by f in s.

In principle, such a function f might express totally unrelated propositions in two states s and t. However, we expect our meanings to act in a *uniform* way across different contexts. The idea is that if the propositions $f(s)$ and $f(t)$ differ, the difference should be traceable to the initial difference in information between s and t. Once the information gap between s and t is filled, the difference between the two propositions should also vanish. This intuition is formalized by the compatibility condition specified below, which requires that when t is an enhancement of s, the proposition $f(t)$ expressed in t should coincide with the restriction of the proposition $f(s)$ to t.

Definition 14 (Compatibility condition)**.**
A function f which associates to any discourse context s a proposition $f(s) \in \Pi_s$ is called *compatible* just in case whenever $t \subseteq s$, $f(t) = f(s)\restriction t$.

We can obtain an intuition by looking at figure 4. Here $s = \{w_1, w_2, w_3, w_4\}$ and $t = \{w_1, w_2, w_3\}$. Suppose the proposition $f(s)$ expressed by a function f on s is the one depicted in figure (a). Then in order for f to be compatible, the proposition $f(t)$ expressed by f on t should be the one depicted in

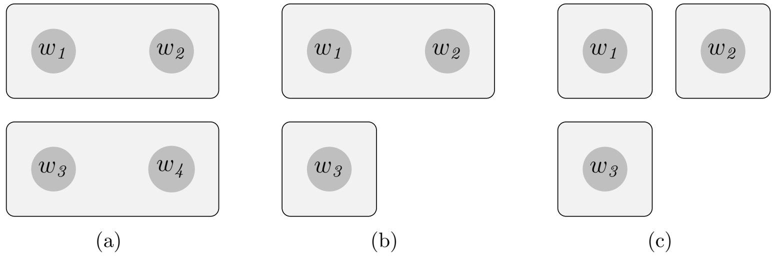


Figure 4: Illustrating the compatibility condition.

figure (b), which is obtained by restricting the proposition in (a) to t, and not, say, the one depicted in figure (c).

Definition 15 (Meanings)**.**

- A *meaning* is a compatible function that maps every discourse context s to a proposition over s.
- The set of all meanings is denoted by $\mathcal{M}$.

We say that a meaning is *informative* in case it has the potential to provide information, that is, if in some discourse contexts it expresses an informative proposition. Similarly, we say that a meaning is *inquisitive* if it has the potential to request information, that is, if in some contexts it expresses an inquisitive proposition.

Definition 16 (Informativeness, inquisitiveness)**.** Let f be a meaning.

- We say that f is *informative* just in case there is a context s such that the proposition $f(s)$ expressed by f in s is informative in s.
- We say that f is *inquisitive* just in case there is a context s such that the proposition $f(s)$ expressed by f in s is inquisitive in s.

Meanings can be ordered in terms of the strength of the propositions they express: we will say that a meaning f *entails* a meaning g in case on any state s, the proposition $f(s)$ entails the proposition $g(s)$.

Definition 17 (Entailment between meanings).
If f and g are meanings, we say that f entails g, in symbols $f \leq g$, just in case $f(s) \subseteq g(s)$ for any context s.

Now, recall that any state s is a substate of the ignorant state ω consisting of all worlds. Thus, if f is a meaning, the compatibility condition yields that for all states s:

$$f(s) = f(\omega)\restriction s$$

This shows that every meaning f is fully determined by a unique proposition, namely the proposition $f(\omega)$ that it expresses in the ignorant state. And vice versa, any proposition A uniquely determines a meaning f_A, which associates to each context s the proposition:

$$f_A(s) = A\restriction s$$

We have thus reached the following conclusion.

Fact 1.
There is a one-to-one correspondence between meanings and propositions.

The meaning of a sentence can thus be given by equipping it with a unique, *absolute* proposition A. The proposition expressed in a particular context s will then be obtained by restricting A to s. But there is more: the following facts ensure that all the properties of meanings we have seen so far, as well as the entailment ordering between them, can be recast in terms of properties of the corresponding propositions and of the entailment ordering between them.

Fact 2. Let f be a meaning.

- f is informative iff the proposition $f(\omega)$ is informative in ω.
- f is inquisitive iff the proposition $f(\omega)$ is inquisitive in ω.

Fact 3. For any two meanings f and g:

$$f \leq g \iff f(\omega) \subseteq g(\omega)$$

Combining facts 1 and 3 we obtain the following result.

Fact 4. The space $\langle \mathcal{M}, \leq \rangle$ of meanings ordered by entailment and the space $\langle \Pi, \subseteq \rangle$ of propositions ordered by entailment are isomorphic.

In the next section, we turn to an investigation of the algebraic structure of the space of proposition. The above result guarantees that the results of this investigation will directly pertain to the structure of meanings as well.

3 Inquisitive algebra

In this section we will investigate the algebraic structure of the space $\langle \Pi, \subseteq \rangle$ of propositions ordered by entailment. This is of course interesting in its own right, but it will also play a crucial role in defining a concrete inquisitive semantics for the language of first-order logic, which we will turn to in the next section. The algebraic results to be presented here will suggest a particular way to deal with connectives and quantifiers. For instance, conjunction will be taken to behave semantically as a *meet* operator, yielding the greatest lower bound of the propositions expressed by its constituents, and other connectives and quantifiers will be associated with other basic algebraic operations, just as in classical logic.

To illustrate our approach, we will first briefly review the algebraic perspective on classical logic. After that, we will turn our attention to the algebra of propositions in inquisitive semantics.[3]

3.1 The algebraic perspective on classical logic

In the classical setting a proposition A is simply a set of possible worlds, which embodies the information that the actual world is located in A. Given this way of thinking about propositions, there is a natural *entailment order* between them: one proposition A entails another proposition B iff A is at least as informative as B, i.e., iff in uttering A, a speaker locates the actual world more precisely than in uttering B. This condition is fulfilled just in case $A \subseteq B$. Thus, the space of classical propositions ordered by entailment is the partially ordered set $\langle \wp(\omega), \subseteq \rangle$.

This space is equipped with a rich algebraic structure. To start with, for any set of propositions Σ, there is a unique proposition that (i) entails all the propositions in Σ, and (ii) is entailed by all other propositions that entail all propositions in Σ. This proposition is the *greatest lower bound* of Σ w.r.t. the entailment order, or in algebraic jargon, its *meet*. It amounts to $\bigcap \Sigma$ (given the stipulation that $\bigcap \emptyset = W$). Similarly, every set of propositions Σ also has a unique *least upper bound* w.r.t. the entailment order, which is called its *join*, and amounts to $\bigcup \Sigma$. The existence of meets and joins for arbitrary sets of classical propositions implies that the space of classical propositions ordered by entailment forms a *complete lattice*.

[3] This section is based on Roelofsen (2011). We present the main results here but omit the proofs.

This lattice is *bounded*. That is, it has a bottom element, $\bot := \emptyset$, and a top element, $\top := \omega$. Moreover, for every two propositions A and B, there is a unique weakest proposition C such that $A \cap C$ entails B. This proposition is called the *pseudo-complement of A relative to B*. It is denoted as $A \Rightarrow B$ and in the case of $\langle \wp(\omega), \subseteq \rangle$, it amounts to $\overline{A} \cup B$, where $\overline{A}$ denotes the set-theoretic complement of A, $\omega - A$. Intuitively, the pseudo-complement of A relative to B is the weakest proposition such that if we 'add' it to A, we get a proposition that is at least as strong as B. The existence of relative pseudo-complements implies that $\langle \wp(\omega), \subseteq \rangle$ forms a *Heyting algebra*.

If A is an element of a Heyting algebra, it is customary to refer to the pseudo-complement of A relative to the bottom element of the algebra, $A^* := (A \Rightarrow \bot)$, as the *pseudo-complement* of A. In the case of $\langle \wp(\omega), \subseteq \rangle$, A^* simply amounts to the set-theoretic complement $\overline{A}$ of A. By definition of pseudo-complements, we have that $A \cap A^* = \bot$ for any element A of any Heyting algebra. In the specific case of $\langle \wp(\omega), \subseteq \rangle$, we also always have that $A \cup A^* = \top$. This means that in $\langle \wp(\omega), \subseteq \rangle$, A^* is in fact the *Boolean complement* of A, and that $\langle \wp(\omega), \subseteq \rangle$ forms a *Boolean algebra*, a special kind of Heyting algebra.

Thus, the space of classical propositions is equipped with certain natural operations. Classical first-order logic is obtained by associating these operations with the logical constants. Indeed, the usual definition of truth can be reformulated as a recursive definition of the set $|\varphi|_g$ of models over a domain D in which φ is true relative to an assignment g. The inductive clauses then run as follows:

- $|\neg\varphi|_g = |\varphi|_g^*$
- $|\varphi \wedge \psi|_g = |\varphi|_g \cap |\psi|_g$
- $|\varphi \vee \psi|_g = |\varphi|_g \cup |\psi|_g$
- $|\varphi \to \psi|_g = |\varphi|_g \Rightarrow |\psi|_g$
- $|\forall x.\varphi|_g = \bigcap_{d \in D} |\varphi|_{g[x \mapsto d]}$
- $|\exists x.\varphi|_g = \bigcup_{d \in D} |\varphi|_{g[x \mapsto d]}$

Negation expresses complementation, conjunction and disjunction express binary meet and join, respectively, implication expresses relative pseudo-complementation, and quantified formulas, $\forall x.\varphi$ and $\exists x.\varphi$, express the infinitary meet and join, respectively, of $\{|\varphi|_{g[x \mapsto d]} \mid d \in D\}$.

Notice that everything starts with a certain notion of propositions and a natural entailment order on these propositions. This entailment order, then, gives rise to certain basic operations on propositions, and classical first-order logic is obtained by associating these basic semantic operations with the logical constants.

3.2 Algebraic operations on propositions in InqB

In exactly the same way, we may investigate the algebraic structure of the space of inquisitive propositions in order to determine which operations could be associated with the logical constants in inquisitive semantics. What kind of algebraic operations exist in the space $\langle \Pi, \subseteq \rangle$?

First, as in the classical setting, any set of meanings $\mathcal{F} \subseteq \Pi$ has a unique greatest lower bound (*meet*) and a unique least upper bound (*join*).

Fact 5 (Meet)**.** For any set $\mathcal{F} \subseteq \Pi$, $\bigcap \mathcal{F}$ is in Π and it is the meet of $\mathcal{F}$.[4]

Fact 6 (Join)**.** For any set $\mathcal{F} \subseteq \Pi$, $\bigcup \mathcal{F}$ is in Π and it is the join of $\mathcal{F}$.

The existence of meets and joins for arbitrary sets of propositions implies that $\langle \Pi, \subseteq \rangle$ forms a complete lattice. And again, this lattice is bounded, i.e., there is a bottom element, $\bot := \{\emptyset\}$, and a top element, $\top := \wp(\omega)$. Finally, as in the classical setting, for every two propositions A and B, there is a unique weakest proposition C such that $A \cap C$ entails B. Recall that this proposition, which is called the pseudo-complement of A relative to B, can be characterized intuitively as the weakest proposition such that if we add it to A, we get a proposition that is at least as strong as B.

Definition 18. For any two propositions A and B:

$$A \Rightarrow B \;:=\; \{\alpha \mid \text{for every } \beta \subseteq \alpha, \text{if } \beta \in A \text{ then } \beta \in B\}$$

Fact 7 (Relative pseudo-complement)**.**
For any $A, B \in \Pi$, $A \Rightarrow B$ is the pseudo-complement of A relative to B.

The existence of relative pseudo-complements implies that $\langle \Pi, \subseteq \rangle$ forms a Heyting algebra. This simple fact will be very useful in the investigation of the logic of our system (see section 5), since it immediately yields the fact

[4] Given the convention that $\bigcap \emptyset = \wp(\omega)$.

that inquisitive logic is an extension of intuitionistic logic. Recall that in a Heyting algebra, $A^* := (A \Rightarrow \bot)$ is referred to as the *pseudo-complement* of A. In the specific case of $\langle \Pi, \subseteq \rangle$, pseudo-complements can be characterized as follows.

Fact 8 (Pseudo-complement)**.** For any proposition $A \in \Pi$:

$$A^* = \{\beta \mid \beta \cap \alpha = \emptyset \text{ for all } \alpha \in A\} = \wp(\overline{\bigcup A})$$

Thus, A^* consists of all states that are disjoint from any element of A. This means that a piece of information settles A^* just in case it locates the actual world outside $\bigcup A$.

So far, then, everything works out just as in the classical setting. However, unlike in the classical setting, the pseudo-complement of a proposition is not always its *Boolean* complement. In fact, most propositions in $\langle \Pi, \subseteq \rangle$ do not have a Boolean complement at all. To see this, suppose that A and B are Boolean complements. This means that (i) $A \cap B = \bot$ and (ii) $A \cup B = \top$. Since $\top = \wp(\omega)$, condition (ii) can only be fulfilled if ω is contained in either A or B. Suppose $\omega \in A$. Then, since A is downward closed, $A = \wp(\omega) = \top$. But then, in order to satisfy condition (i), we must have that $B = \{\emptyset\} = \bot$. So the only two elements of our algebra that have a Boolean complement are $\top$ and $\bot$. Hence, the space $\langle \Pi, \subseteq \rangle$ of inquisitive propositions does not form a Boolean algebra.

Thus, starting with a new notion of propositions and a suitable entailment order on these propositions that takes both informative and inquisitive content into account, we have established an algebraic structure with two extremal elements and three basic operations, *meet*, *join*, and *relative pseudo-complementation*. The only algebraic difference with respect to the classical setting is that, except for the extremal elements, inquisitive propositions do not have Boolean complements. However, as in the classical setting, every proposition does have a pseudo-complement. This algebraic result gives rise to an inquisitive semantics for a first-order language, to which we turn now.

4 Inquisitive semantics

In this section we define an inquisitive semantics for a first-order language, motivated by the algebraic results presented in the previous section. We will investigate the main properties of the system, and illustrate it with a range of examples.

4.1 Inquisitive semantics for a first-order language

We will consider a standard first-order language $\mathcal{L}$, with $\bot$, $\vee$, $\wedge$, $\rightarrow$, $\exists$, and $\forall$ as its basic logical constants. We will use $\neg\varphi$ as an abbreviation of $\varphi \rightarrow \bot$. We will also use $!\varphi$ as an abbreviation of $\neg\neg\varphi$ and $?\varphi$ as an abbreviation of $\varphi \vee \neg\varphi$. We refer to ! as the *declarative* operator and to ? as the *interrogative* operator. The precise role of these operators in the system will become clear below, especially in section 4.6.

In order to simplify matters, we consider a fixed domain D and a fixed interpretation of constants and function symbols: that is, we restrict our attention to the case in which the domain of discourse and the reference of proper names are common knowledge among the discourse participants, and the only uncertainty concerns the extension of predicates and relations.

Formally, we consider a fixed domain structure $\mathbb{D} = (D, I_{\mathbb{D}})$ which consists of a domain D and an interpretation function $I_{\mathbb{D}}$ that maps every individual constant c to an object in D and every $n-$ary function symbol f to a function from D^n to D. Our logical space consists of first-order models based on $\mathbb{D}$.

Definition 19 ($\mathbb{D}$-worlds)**.** A $\mathbb{D}-$*world* is a structure $w = (\mathbb{D}, I_w)$, where I_w is a function interpreting each $n-$ary relation symbol R as a relation $I_w(R) \subseteq D^n$. The set of all $\mathbb{D}-$worlds is denoted $\omega_{\mathbb{D}}$.

Unless specified otherwise, the structure $\mathbb{D}$ will be considered fixed throughout our discussion and we shall drop reference to it whenever possible. In order not to have assignments in the way all the time, we will assume that for any $d \in D$, the language $\mathcal{L}$ contains an individual constant $\overline{d}$ such that $I_{\mathbb{D}}(\overline{d}) = d$: if this is not the case, we simply expand the language by adding new constants, and we expand $I_{\mathbb{D}}$ accordingly. In this way we can define our semantics for sentences only, and we can do without assignments altogether. This move is not essential, but it simplifies both notation and terminology.

If φ is a sentence of $\mathcal{L}$, we denote by $|\varphi|$ the set of all worlds where φ is classically true; we call $|\varphi|$ the *truth-set* of φ.

Definition 20 (Truth-set)**.** The truth set $|\varphi|$ of a formula φ is the set of worlds where φ is classically true.

We are now ready to recursively associate a proposition to each sentence of our first-order language. We start with atomic sentences: the proposition expressed by an atomic sentence $R(t_1, \dots, t_n)$ is defined as the set of all states that consist exclusively of worlds where $R(t_1, \dots, t_n)$ is true. This means that in uttering an atomic sentence, a speaker provides the information that the actual world is one where that sentence is true, and does not request any further information. Thus, atomic sentences are treated just as in the classical setting. The inductive clauses of the semantics are driven by the algebraic results established in the previous section. That is, the logical constants are taken to express the fundamental operations that we identified on our space of propositions.

Definition 21 (Inquisitive semantics for a first-order language)**.**

1. $[R(t_1, \dots, t_n)] := \wp(|R(t_1, \dots, t_n)|)$
2. $[\bot] := \{\emptyset\}$
3. $[\varphi \wedge \psi] := [\varphi] \cap [\psi]$
4. $[\varphi \vee \psi] := [\varphi] \cup [\psi]$
5. $[\varphi \to \psi] := [\varphi] \Rightarrow [\psi]$
6. $[\forall x.\varphi(x)] := \bigcap_{d \in D} [\varphi(\overline{d})]$
7. $[\exists x.\varphi(x)] := \bigcup_{d \in D} [\varphi(\overline{d})]$

We refer to this first-order system as $\mathsf{Inq}_{\mathrm{B}}$, where B stands for *basic*. We refer to $[\varphi]$ as the *proposition expressed* by φ, and to the elements of $[\varphi]$ as the *possibilities* for φ. The clauses of $\mathsf{Inq}_{\mathrm{B}}$ constitute a proper inquisitive semantics in the sense that they indeed associate every sentence $\varphi \in \mathcal{L}$ with a proposition in the sense of section 2.2.

Fact 9 (Suitability of the semantics)**.**
For every $\varphi \in \mathcal{L}$, $[\varphi] \in \Pi$.

Since negation is defined as an abbreviation, its semantic behavior is determined by that of the basic connectives. The derived clause for negation is given below. Notice that $[\neg\varphi] = [\varphi]^*$, which means that negation expresses the pseudo-complementation operation.

Fact 10 (Derived clause for negation)**.**

- $[\neg\varphi] \;=\; [\varphi \to \bot] \;=\; [\varphi] \Rightarrow \{\emptyset\} \;=\; [\varphi]^* \;=\; \wp(\overline{\bigcup[\varphi]})$

The *informative content* of a sentence φ, $\mathsf{info}(\varphi)$, is defined as the informative content of the proposition it expresses, which amounts to $\bigcup[\varphi]$.

Definition 22 (Informative content)**.** For every $\varphi \in \mathcal{L}$: $\mathsf{info}(\varphi) := \bigcup[\varphi]$

A sentence φ *entails* another sentence ψ just in case the proposition expressed by φ entails the proposition expressed by ψ, and φ and ψ are *equivalent*, $\varphi \equiv \psi$, just in case they express exactly the same proposition.

Definition 23 (Entailment and equivalence)**.** Let $\varphi, \psi \in \mathcal{L}$. Then:

- φ entails ψ just in case $[\varphi] \subseteq [\psi]$
- φ and ψ are equivalent, $\varphi \equiv \psi$, just in case $[\varphi] = [\psi]$

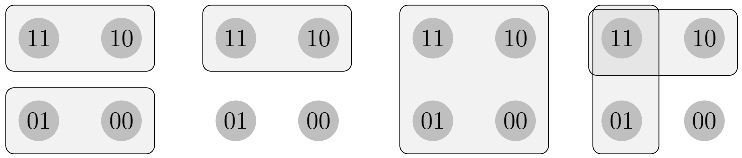
4.2 Semantic categories

We call a sentence *informative* just in case it has an informative meaning, i.e., if it has the potential to express an informative proposition in some state, and *inquisitive* just in case it has an inquisitive meaning, i.e. if it has the potential to express an inquisitive proposition in some state. Fact 4 ensures that these properties can be recast in terms of properties of the proposition $[\varphi]$ as follows.

Definition 24 (Informativeness, inquisitiveness)**.**

- A sentence φ is informative if $\mathsf{info}(\varphi) \neq \omega$.
- A sentence φ is inquisitive if $\mathsf{info}(\varphi) \notin [\varphi]$.

These semantic properties allow us to distinguish several classes of sentences. First, we have sentences that are non-inquisitive, lacking the potential to request information. The meaning of such sentences consists exclusively in their informative potential, which means that, if their utterance in a context has any effect at all, then what it does is to provide information. We call such sentences *assertions*. Symmetrically, there are sentences that are non-informative, lacking the potential to provide information. Their meaning consists exclusively in their inquisitive potential, which means that, if their utterance in a context has any effect at all, then what it does is to request information. Such sentences are called *questions*. Thirdly, there are sentences that lack both informative and inquisitive potential. When uttered



(a) Question (b) Assertion (c) Tautology (d) Hybrid

Figure 5: Semantic categories.

in a context, these sentences never have any effect at all. They are thus insignificant, and we call them *tautologies*. Finally, there are sentences which are both informative and inquisitive. Such sentences, which are capable both of providing and of requesting information, are called *hybrids*.

Definition 25 (Semantics categories)**.** We say that a sentence φ is:

- an *assertion* iff it is non-inquisitive;
- a *question* iff it is non-informative;
- a *tautology* iff it is neither informative nor inquisitive;
- a *hybrid* iff it is both informative and inquisitive.

These four semantic categories are exemplified in figure 5. Spelling out what it means to be non-informative and non-inquisitive we obtain the following direct characterization of questions, assertions and tautologies.

Fact 11 (Direct characterization of questions, assertions and tautologies)**.**

- φ is an assertion $\iff \mathsf{info}(\varphi) \in [\varphi]$.
- φ is a question $\iff \mathsf{info}(\varphi) = \omega$.
- φ is a tautology $\iff \omega \in [\varphi]$.

Notice that if $\mathsf{info}(\varphi) \in [\varphi]$ then, since $\mathsf{info}(\varphi) = \bigcup[\varphi]$, $\mathsf{info}(\varphi)$ must be the greatest element of $[\varphi]$. Vice versa, if $[\varphi]$ has a greatest element, it is easy to see that this greatest element must be precisely $\mathsf{info}(\varphi)$. We have thus established the following fact, which makes it particularly easy to say whether a sentence is an assertion, given a visualization of the proposition that it expresses.

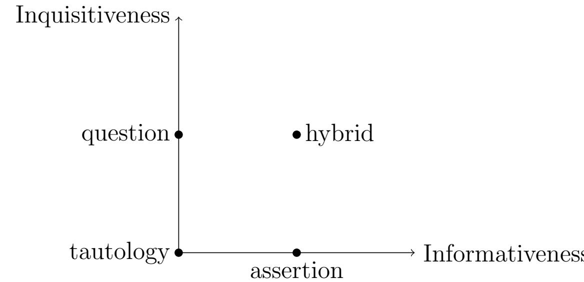


Figure 6: Questions, assertions, tautologies, and hybrids in a two-dimensional space.

Fact 12 (Alternative characterization of assertions)**.**

$$\begin{aligned} \varphi \text{ is an assertion} \quad &\iff \quad [\varphi] = \wp(\mathsf{info}(\varphi)) \\ &\iff \quad [\varphi] \text{ has a greatest element.} \end{aligned}$$

We can visualize sentences as inhabiting a two-dimensional space, as depicted in figure 6, arranged according to the strength of their informative and inquisitive content. The informative axis, where the inquisitive component is trivial, is inhabited by assertions, which are non-inquisitive. The inquisitive axis, where the informative component is trivial, is inhabited by questions, which are non-informative. The ‘zero-point’ of the space, where both components are trivial, is inhabited by tautologies, which are neither informative nor inquisitive. The rest of the space, where both components are non-trivial, is inhabited by hybrids, which are both informative and inquisitive.

4.3 Information and truth

Let us now look more closely at how informative content is treated in $\mathsf{Inq_B}$. Recall that $\mathsf{info}(\varphi)$ is defined as $\bigcup[\varphi]$. Thus, $\mathsf{info}(\varphi)$ is a state. In uttering the sentence φ, a speaker provides the information that the actual world lies in this state. In classical logic, the informative content of a sentence φ is also embodied by a state, namely the truth-set $|\varphi|$, consisting of all worlds where φ is true. Now, the question that naturally arises is how these notions of informative content relate to each other. The following fact answers this question, establishing that the two notions always coincide.

Fact 13 (The treatment of information in $\mathsf{Inq}_{\mathrm{B}}$ is classical).
For any sentence φ, $\mathsf{info}(\varphi) = |\varphi|$.

This shows that $\mathsf{Inq}_{\mathrm{B}}$ preserves the classical treatment of informative content. $\mathsf{Inq}_{\mathrm{B}}$ only differs from classical logic in that it captures an additional aspect of meaning, namely inquisitive content.

Notice that in combination with facts 11 and 12, fact 13 entails the following characterization of assertions and questions in terms of classical truth.

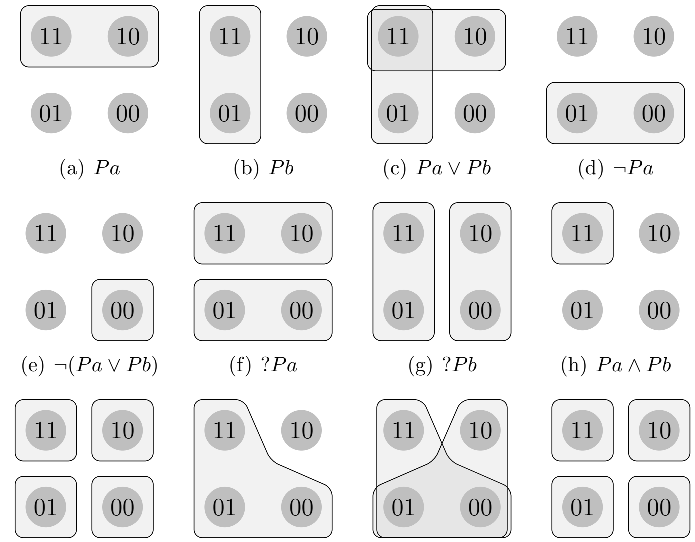
Fact 14 (Questions, assertions, and classical truth).

- φ is a question $\iff |\varphi| = \omega$
- φ is an assertion $\iff |\varphi| \in [\varphi] \iff [\varphi] = \wp(|\varphi|)$

Thus, questions are sentences that are classically true at any world. The proposition expressed by an assertion φ always amounts to $\wp(|\varphi|)$, which means that in uttering an assertion φ, a speaker is taken to provide the information that φ is true, and not to request any further information. Thus, assertions behave in $\mathsf{Inq}_{\mathrm{B}}$ exactly as they do in classical logic. We will see in section 5 that this classical behavior of assertions is also reflected in the logic that $\mathsf{Inq}_{\mathrm{B}}$ gives rise to.

4.4 Examples

In this section we will consider some simple sentences and examine the propositions that they express. We consider a language with just one unary predicate symbol, P, and two individual constants, a and b. Accordingly, we assume that the domain of discourse consists of just two objects, denoted by a and b, respectively. Our logical space, then, consists of four worlds, one in which both Pa and Pb are true, one in which Pa is true but Pb is false, one in which Pb is true but Pa is false, and one in which neither Pa nor Pb is true. In figure 7, these worlds are labeled 11, 10, 01, and 00, respectively. As usual, in order to keep the pictures orderly we display only the maximal elements of a proposition.



(a) Pa (b) Pb (c) $Pa \vee Pb$ (d) $\neg Pa$

(e) $\neg(Pa \vee Pb)$ (f) $?Pa$ (g) $?Pb$ (h) $Pa \wedge Pb$

(i) $?Pa \wedge ?Pb$ (j) $Pa \to Pb$ (k) $Pa \to ?Pb$ (l) $\forall x.?Px$

Figure 7: The propositions expressed by some simple sentences.

Atomic sentences. Let us first consider the proposition expressed by one of the two atomic sentences in our language, Pa. According to the clause for atomic sentences, $[Pa]$ consists of all states s such that every world in s makes Pa true: these states are $\{11, 10\}$, $\{11\}$, $\{10\}$, $\emptyset$. Thus, $[Pa]$ has a greatest element, namely the state $\{11, 10\}$ depicted in figure 7(a). Fact 12 therefore ensures that Pa is an assertion, and thus, according to fact 14, it behaves just like in the classical setting, providing the information that Pa is true and not requesting any further information.

Analogously, the other atomic sentence in our language, Pb, is an assertion which provides the information that Pb is true. The proposition expressed by Pb is depicted in figure 7(b).

Disjunction. Next, we consider the disjunction $Pa \vee Pb$. According to the clause for disjunction, $[Pa \vee Pb]$ consists of those states that are either in $[Pa]$ or in $[Pb]$. These are $\{11, 10\}$, $\{11, 01\}$, $\{11\}$, $\{10\}$, $\{01\}$, and $\emptyset$.

Since $|Pa \vee Pb| = \{11, 10, 01\} \neq \omega$, the sentence $Pa \vee Pb$ is informative. More precisely, it provides the information that either Pa or Pb is true. However, unlike in the case of atomic sentences, in this case there is no greatest possibility that includes all the others. Instead, there are *two* maximal possibilities, $\{11, 10\}$ and $\{11, 01\}$, which together contain all the others. Thus, according to fact 12, $Pa \vee Pb$ is not an assertion, but it is inquisitive. In order to settle the issue raised by $Pa \vee Pb$, one has to establish either a state $s \subseteq \{11, 10\} = |Pa|$, or a state $s \subseteq \{11, 01\} = |Pb|$. In the former case, one establishes that Pa is true; in the latter, one establishes that Pb is true.

Thus, in inquisitive semantics the formula $Pa \vee Pb$ is a hybrid, which provides the information that at least one of the disjuncts is true, and requests enough information to establish for at least one of the disjuncts that it is true.

Negation. Next, we turn to negation. According to the derived clause for negation, $[\neg Pa]$ consists of all states s such that s does not have any world in common with any state in $[Pa]$. Thus, $[\neg Pa]$ consists of all states that do not contain the worlds 11 and 10, which are $\{01, 00\}$, $\{01\}$, $\{00\}$, and $\emptyset$, as depicted in figure 7(d). Since this set of states has a greatest element, namely $\{01, 00\}$, fact 12 ensures that $\neg Pa$ is an assertion. And since the behavior of assertions is classical, $\neg Pa$ simply provides the information that Pa is false, without requesting any further information.

Now let us consider the negation of a non-atomic sentence, $\neg(Pa \vee Pb)$. According to the clause for negation, $[\neg(Pa \vee Pb)]$ consists of all states which do not have any world in common with any state in $[Pa \vee Pb]$. Thus, $[\neg(Pa \vee Pb)]$ consists of all states that do not contain the worlds 11, 10, and 01, which are $\{00\}$ and $\emptyset$, as depicted in figure 7(e). Again, there is a unique maximal possibility, namely $\{00\} = |\neg(Pa \vee Pb)|$. Thus, $\neg(Pa \vee Pb)$ is an assertion, which behaves classically, providing the information that both Pa and Pb are false, and not requesting any further information.

These cases of negative sentences exemplify a general fact which is apparent from the semantic clause for negation (fact 10): for any sentence φ, the proposition $[\neg\varphi]$ always contains a greatest element, namely $\overline{\bigcup[\varphi]} = \overline{|\varphi|}$. Thus, a negative formula $\neg\varphi$ is always an assertion, which provides the information that φ is false. We will come back to this property of negation in section 4.5.

Interrogatives. Let us now consider another simple sentence involving both negation and disjunction, the interrogative $?Pa$, defined as $Pa \vee \neg Pa$. We have already seen what $[Pa]$ and $[\neg Pa]$ look like. According to the clause for disjunction, $[?Pa] = [Pa \vee \neg Pa]$ consists of all states that are either in $[Pa]$ or in $[\neg Pa]$. These states are $\{11, 10\}$, $\{01, 00\}$, and all substates thereof, as depicted in figure 7(f). Since $|?Pa| = |Pa \vee \neg Pa| = \omega$, the sentence $?Pa$ is not informative, that is, it is a question. However, since $[?Pa]$ does not have a unique greatest element, it is inquisitive. In order to settle the issue raised by $?Pa$, one has to establish either a state $s \subseteq \{11, 10\} = |Pa|$, or a state $s \subseteq \{01, 00\} = |\neg Pa|$. In the former case, one establishes that Pa is true; in the latter, one establishes that Pa is false. Hence, in order to settle the issue raised by $?Pa$, one has to establish whether Pa is true. Thus, while $?Pa$ is a shorthand for $Pa \vee \neg Pa$, perhaps the most famous classical tautology, this formula is not a tautology in $\mathsf{Inq}_{\mathrm{B}}$: instead, it correponds to the polar question "whether Pa". Analogously, $?Pb$, depicted in figure 7(g), correponds to the polar question "whether Pb".

Conjunction. Now let us consider conjunction. First, let us look at the conjunction of our two atomic sentences, Pa and Pb. According to the clause for conjunction, $[Pa \wedge Pb]$ consists of those states that are both in $[Pa]$ and in $[Pb]$. These are $\{11\}$ and $\emptyset$. Thus, $[Pa \wedge Pb]$ has a greatest element, namely $\{11\}$, and accordingly $Pa \wedge Pb$ is an assertion which, just like in the classical case, provides the information that both Pa and Pb are true.

Now let us look at the conjunction of two complex sentences, the polar questions $?Pa$ and $?Pb$. As depicted in figure 7(i), the proposition $[?Pa \wedge ?Pb]$ consists of the states $\{11\}$, $\{10\}$, $\{01\}$, $\{00\}$ and $\emptyset$. Since $|?Pa \wedge ?Pb| = \omega$, our conjunction is a question. Moreover, since there is no unique maximal possibility, this question is inquisitive. In order to settle the issue it raises, one has to provide enough information to establish at least one of $Pa \wedge Pb$, $Pa \wedge \neg Pb$, $\neg Pa \wedge Pb$, $\neg Pa \wedge \neg Pb$. Thus, our conjunction is a question which requests enough information to settle both the issue whether Pa, raised by $?Pa$, and the issue whether Pb, raised by $?Pb$.

These two cases of conjunctive formulas exemplify a general fact: if φ and ψ are assertions, then the conjunction $\varphi \wedge \psi$ is itself an assertion which provides both the information provided by φ and the information provided by ψ; and if φ and ψ are questions, then the conjunction $\varphi \wedge \psi$ is itself a question, which requests the information needed to settle both the issue raised by φ and the issue raised by ψ.

Implication. Next, let us consider implication. Again, we will first consider a simple case, $Pa \to Pb$, where both the antecedent and the consequent are atomic. According to the clause for implication, $[Pa \to Pb]$ consists of all states s such that every substate $t \subseteq s$ that is in $[Pa]$ is also in $[Pb]$. These are precisely the states s such that $s \subseteq \{11, 01, 00\}$, as depicted in figure 7(j). So, $[Pa \to Pb]$ has a greatest element, $|Pa \to Pb| = \{11, 01, 00\}$, which means that the implication $Pa \to Pb$ is an assertion which, just like in the classical setting, provides the information that if Pa is true, then so is Pb.

Now let us consider a more complex case, $Pa \to ?Pb$, where the consequent is the polar question "whether Pb". As depicted in figure 7(k), the proposition $[Pa \to ?Pb]$ consists of the states $\{11, 01, 00\}$, $\{10, 01, 00\}$, and all substates thereof. Since $|Pa \to ?Pb| = \omega$, our implication is a question. Moreover, since there is no unique greatest possibility, this question is inquisitive. In order to settle the issue it raises, one must either establish a state $s \subseteq \{11, 01, 00\} = |Pa \to Pb|$, or a state $s \subseteq \{10, 01, 00\} = |Pa \to \neg Pb|$. In the former case one establishes that if Pa is true then so is Pb; in the latter case, that if Pa is true then Pb is false. So, in $\mathsf{Inq}_{\mathrm{B}}$ the sentence $Pa \to ?Pb$ is a question which requests the information needed to establish whether Pb is the case under the assumption that Pa is the case.

Again, these two cases of conditional formulas exemplify a general feature of $\mathsf{Inq}_{\mathrm{B}}$: if ψ is an assertion, then $\varphi \to \psi$ is an assertion which provides the information that if φ is true, then so is ψ; and if ψ is a question, then $\varphi \to \psi$ is a question which requests the information needed to settle the issue raised by ψ assuming the information provided by φ and a resolution of the issue raised by φ.

Quantification. Finally, let us consider existential and universal quantification. As usual, existential quantification behaves essentially like disjunction and universal quantification behaves essentially like conjunction. In fact, since our current domain of discourse consists of only two objects, denoted by a and b, respectively, $\exists x.Px$ expresses exactly the same proposition as $Pa \vee Pb$, depicted in figure 7(c), and $\forall x.Px$ expresses exactly the same proposition as $Pa \wedge Pb$, depicted in figure 7(h). Finally, consider the proposition expressed by $\forall x.?Px$, depicted in figure 9(b). Notice that this

proposition induces a partition on the logical space, where each block of the partition consists of worlds that agree on the extension of P. Thus, $\forall x.?Px$ is a question that asks for an exhaustive specification of the objects that have the property P. This concludes our illustration of the behavior or the logical constants in $\mathsf{Inq}_{\mathrm{B}}$.

4.5 Syntactic properties of questions and assertions

Assertions were defined in section 4.2 as sentences whose meaning consists exclusively in their informative potential, and questions as sentences whose meaning consists exclusively in their inquisitive potential. Notice that these characterizations are *semantic* in nature. In this section we provide *syntactic* conditions for sentences to be assertions or questions.

Let us start by examining assertions. The following fact provides some sufficient syntactic conditions, which generalize the particular observations made in the previous section.

Fact 15 (Sufficient conditions for assertionhood)**.**

1. An atomic sentence $R(t_1, \ldots, t_n)$ is an assertion;
2. $\bot$ is an assertion;
3. if φ and ψ are assertions, then so is $\varphi \wedge \psi$;
4. if ψ is an assertion, then so is $\varphi \to \psi$ for any sentence φ;
5. if $\varphi(\overline{d})$ is an assertion for all $d \in D$, then so is $\forall x \varphi(x)$.

This fact immediately yields the following corollary, which shows that disjunction and the existential quantifier are the only sources of inquisitiveness in our logical language.

Corollary 1 (Sources of inquisitiveness)**.**
Any sentence that does not contain $\vee$ or $\exists$ is an assertion.

Also, since a negation $\neg\varphi$ is an abbreviation for $\varphi \to \bot$, items 2 and 4 combined yield the following corollary.

Corollary 2 (Negations are assertions)**.**
$\neg\varphi$ is an assertion for any φ.

Now let us turn to syntactic conditions for being a question, which again generalize our particular observations in the previous section.

Fact 16 (Sufficient conditions for questionhood)**.**

1. Any classical tautology is a question;
2. if φ and ψ are questions, so is $\varphi \wedge \psi$;
3. if ψ is a question, then for any φ so are $\varphi \vee \psi$ and $\varphi \rightarrow \psi$;
4. if $\varphi(\overline{d})$ is a question for all $d \in D$, then so is $\forall x\varphi(x)$;
5. if $\varphi(\overline{d})$ is a question for some $d \in D$, then so is $\exists x\varphi(x)$.

4.6 Projection operators

We proposed in section 4.2 to regard sentences of $\mathsf{Inq}_{\mathrm{B}}$ as inhabiting a two dimensional space, where assertions lie on the horizontal axis and questions on the vertical axis. A natural question that arises, then, is whether we can define *projection operators* on this space, i.e., whether there are natural ways to turn any given sentence into an assertion, or into a question.

Suppose we add an operator $\mathcal{A}$ to our language, which is intended to behave as a non-inquisitive projection operator, turning every sentence into an assertion. How should the semantic contribution of $\mathcal{A}$ be defined in order for it to behave as a proper non-inquisitive projection operator? First of all, for any φ, $\mathcal{A}\varphi$ should be an assertion. Moreover, it is natural to require that, while trivializing inquisitive content, A should preserve informative content, that is, $\mathcal{A}\varphi$ should have the same informative content as φ.

Definition 26 (Non-inquisitive projection operator)**.**
We call an operator $\mathcal{A}$ a *non-inquisitive projection operator* just in case for any φ:

- $\mathcal{A}\varphi$ is an assertion
- $\mathsf{info}(\mathcal{A}\varphi) = \mathsf{info}(\varphi)$

Now, in section 4.2 we saw that the meaning of an assertion φ is completely determined by its informative component: if φ is an assertion, we must have that $[\varphi] = \wp(\mathsf{info}(\varphi))$. This means that the semantic behavior associated with a non-inquisitive projection operator is uniquely determined.

Fact 17 (Uniqueness of the non-inquisitive projection operator)**.**
$\mathcal{A}$ is a non-inquisitive projection operator if and only if $[\mathcal{A}\varphi] = \wp(\mathsf{info}(\varphi))$.

Now, recall that the declarative $!\varphi$ was defined as an abbreviation for $\neg\neg\varphi$. According to corollary 2, $!\varphi$ is an assertion for any φ. Moreover, using fact 13 we have that $\mathsf{info}(!\varphi) = |!\varphi| = |\neg\neg\varphi| = |\varphi| = \mathsf{info}(\varphi)$, which shows that ! preserves informative content. This means that the declarative operator ! is a non-inquisitive projection operator. Moreover, the previous fact guarantees that any non-inquisitive projection operator must be equivalent with it.

Fact 18 (! is the non-inquisitive projection operator)**.**

- The declarative operator ! is a non-inquisitive projection operator;
- If $\mathcal{A}$ is a non-inquisitive projection operator, then $[\mathcal{A}\varphi] = [!\varphi]$ for all φ.

Identifying what requirements we should place on a non-informative projection operator $\mathcal{Q}$ is less straightforward. Obviously, we should require $\mathcal{Q}$ to trivialize informative content; that is, $\mathcal{Q}$ should turn any formula φ into a question $\mathcal{Q}\varphi$. But we cannot just require $\mathcal{Q}$ to preserve inquisitive content: for, if φ and $\mathcal{Q}\varphi$ do not have the same informative content, then the inquisitive content of φ and the inquisitive content of $\mathcal{Q}\varphi$ are issues over different states, and therefore they are necessarily different objects.

To solve this problem, we will associate to any formula an issue $D(\varphi)$ over ω, no matter what the informative content of φ is. This issue requests enough information to either settle the issue $[\varphi]$, thus locating the actual world in a possibility for φ, or to reject the informative content of φ altogether, thus locating the actual world outside of any possibility for φ and making the issue $[\varphi]$ insubstantial.

Definition 27 (Settling, contradicting, and deciding on a proposition)**.**
Let s be an information state and A a proposition. Then we say that:

- s *settles* A in case $s \in A$;
- s *contradicts* A in case $s \cap \mathsf{info}(A) = \emptyset$;
- s *decides* on A in case s settles A or s contradicts A.

Definition 28 (Decision set)**.**
The *decision set* $D(\varphi)$ of a sentence φ is the set of states that decide on $[\varphi]$.

The decision set of a sentence can be characterized explicitly as follows.

Fact 19. For any φ, $D(\varphi) = [\varphi] \cup [\varphi]^*$

Notice that the decision set $D(\varphi)$ of a sentence φ is always an issue over ω, no matter what the informative content of ω is. Therefore, we can require of a non-informative projection operator $\mathcal{Q}$ that it preserve the decision set of the sentence it applies to.

Definition 29 (Non-informative projection operator)**.**
We call an operator $\mathcal{Q}$ a *non-informative projection operator* just in case for any φ:

- $\mathcal{Q}\varphi$ is a question;
- $D(\mathcal{Q}\varphi) = D(\varphi)$.

Now suppose that Q is a non-informative projection operator. Then for any φ, $\mathcal{Q}\varphi$ should be a question, which means that the informative content of $\mathcal{Q}\varphi$ should be ω. But then $[\mathcal{Q}\varphi]^* = \wp(\overline{\mathsf{info}(\mathcal{Q}\varphi)}) = \{\emptyset\}$, and therefore $D(\mathcal{Q}\varphi) = [\mathcal{Q}\varphi] \cup [\mathcal{Q}\varphi]^* = [\mathcal{Q}\varphi]$. But since $\mathcal{Q}$ should preserve the decision set of φ, we must also have $D(\mathcal{Q}\varphi) = D(\varphi) = [\varphi] \cup [\varphi]^*$. Putting these things together, we obtain that we must have $[\mathcal{Q}\varphi] = [\varphi] \cup [\varphi]^*$. We have thus found that the requirements we placed on $\mathcal{Q}$ uniquely determine its semantic behavior.

Fact 20 (Uniqueness of the non-informative projection operator)**.**
$\mathcal{Q}$ is a non-informative projection operator if and only if $[\mathcal{Q}\varphi] = [\varphi] \cup [\varphi]^*$.

Now recall that the interrogative operator ? was introduced by the convention that $?\varphi$ abbreviates $\varphi \vee \neg\varphi$. Spelling out the semantics of negation and disjunction, we have that $[?\varphi] = [\varphi \vee \neg\varphi] = [\varphi] \cup [\varphi]^*$. Thus, the interrogative operator ? is a non-informative projection operator. Moreover, the previous fact guarantees that any non-informative projection operator must be equivalent with it.

Fact 21 (? is the non-informative projection operator)**.**

- The interrogative operator ? is a non-informative projection operator;
- If $\mathcal{Q}$ is a non-informative projection operator, $[\mathcal{Q}\varphi] = [?\varphi]$ for all φ.

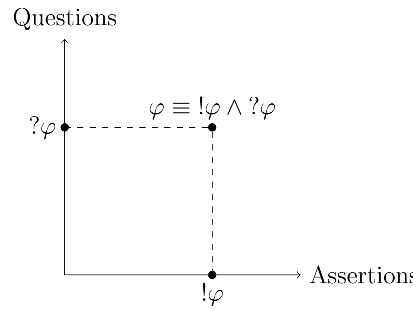


Figure 8: Non-informative and non-inquisitive projections.

Thus, for any sentence φ, the projection operators yield an assertion $!\varphi$ which has the same informative content as φ, and a question $?\varphi$ which has the same decision set as φ. The following fact says that the full meaning of φ can then be reconstructed as the conjunction of these "pure components". Thus, we obtain a representation of the meaning of φ in which the labor is divided between an assertion, that takes care of the informative content of φ, and a question, that takes care of the inquisitive content of φ.

Fact 22 (Division)**.** For any φ, $\varphi \equiv\, !\varphi \wedge ?\varphi$

4.7 Propositions expressed in a state and support

When a sentence φ is uttered in a context s, it expresses a proposition $[\varphi]_s$ which embodies a proposal to enhance the context s in certain ways. As discussed in section 2.3, this proposition is obtained by restricting the absolute proposition $[\varphi]$ expressed by φ to s.

Definition 30 (Proposition expressed by a sentence in a state)**.**
The proposition $[\varphi]_s$ expressed by a sentence φ in a state s is defined as:

$$[\varphi]_s = [\varphi]\restriction s = \{t \subseteq s \mid t \in [\varphi]\}$$

The informative content of this proposition is the set $\mathsf{info}([\varphi]_s) = \bigcup[\varphi]_s$. Using the definition of $[\varphi]_s$ and the fact that $\bigcup[\varphi] = |\varphi|$, it is easy to see that $\mathsf{info}([\varphi]_s) = |\varphi| \cap s$. We denote this set by $|\varphi|_s$ and we call it the *information provided by φ in s.*

Definition 31 (Information provided by a sentence in a state)**.**
The information $|\varphi|_s$ provided by a sentence φ in a state s is the state $|\varphi| \cap s$.

Now, we will say that a sentence φ is informative (resp. inquisitive) in s in case it expresses an informative (resp. inquisitive) proposition in s.

Definition 32 (Informativeness and inquisitiveness in a state)**.**

- φ is *informative* in s iff the proposition $[\varphi]_s$ is informative in s.
- φ is *inquisitive* in s iff the proposition $[\varphi]_s$ is inquisitive in s.

The following fact provides an explicit characterization of informativeness and inquisitiveness of a sentence in a state in terms of the information it provides and the proposition it expresses in that state.

Fact 23 (Informativeness and inquisitiveness in a state)**.**

- φ is *informative* in s iff $|\varphi|_s \neq s$.
- φ is *inquisitive* in s iff $|\varphi|_s \notin [\varphi]_s$.

If φ is neither informative nor inquisitive in s, then an utterance of φ in the context s has no effect at all. In this case, we say that s *supports* φ.

Definition 33 (Support)**.**
A state s supports a sentence φ, in symbols $s \models \varphi$, in case s is neither informative nor inquisitive in s.

Spelling out the definition of informativeness and inquisitiveness in a state, we see that s supports φ if and only if $|\varphi|_s \in [\varphi]_s$ and $|\varphi|_s = s$, that is, if and only if $s \in [\varphi]_s$. And since $[\varphi]_s = \{\alpha \subseteq s \mid \alpha \in [\varphi]\}$, the condition $s \in [\varphi]_s$ is equivalent to $s \in [\varphi]$. Hence, we have found the following tight connection between support and the proposition expressed by a sentence.

Fact 24 (Propositions and support)**.**
For any sentence φ and any state s:

$$s \models \varphi \iff s \in [\varphi]$$

Support is to meanings in $\mathsf{Inq}_{\mathrm{B}}$ what truth is to classical meanings. Indeed, if we write $w \models^{\mathsf{cl}} \varphi$ for "φ is classically true in w", the connection between classical truth and classical meanings can be formulated as follows:

$$w \models^{\mathsf{cl}} \varphi \iff w \in |\varphi|$$

Just like the proposition expressed by φ in classical logic coincides with the set of worlds where φ is true, the proposition expressed by φ in $\mathsf{Inq}_{\mathrm{B}}$ coincides with the set of states where φ is supported. As a consequence, just like classical logic can be characterized by means of a recursive definition of the truth conditions of the sentences in the language, $\mathsf{Inq}_{\mathrm{B}}$ can be characterized by a recursive definition of the support conditions of the sentences in the language. These support conditions are as follows.

Fact 25 (Support)**.**

1. $s \models R(t_1, \dots, t_n)$ iff $s \subseteq |R(t_1, \dots, t_n)|$
2. $s \models \bot$ iff $s = \emptyset$
3. $s \models \varphi \wedge \psi$ iff $s \models \varphi$ and $s \models \psi$
4. $s \models \varphi \vee \psi$ iff $s \models \varphi$ or $s \models \psi$
5. $s \models \varphi \rightarrow \psi$ iff $\forall t \subseteq s$: if $t \models \varphi$ then $t \models \psi$
6. $s \models \forall x \varphi(x)$ iff for all $d \in D$, $s \models \varphi(\overline{d})$
7. $s \models \exists x \varphi(x)$ iff for some $d \in D$, $s \models \varphi(\overline{d})$

In much previous work on inquisitive semantics (e.g. Groenendijk and Roelofsen, 2009; Ciardelli, 2009; Ciardelli and Roelofsen, 2011), support is indeed presented as the basic semantic notion, with propositions and meanings as derived notions. One advantage of this approach is that it parallels the usual presentation of classical logic, with truth as the basic notion. Another advantage is that the support conditions immediately suggests a connection with intuitionistic logic. The set of non-empty states, ordered by the relation $\supseteq$, constitutes a Kripke frame for intuitionisitic logic. The support clauses amount precisely to the usual Kripke semantics for intuitionistic logic on this frame, the particular valuation function being provided by the clause for atoms. This connection is explored in depth in Ciardelli (2009) and Ciardelli and Roelofsen (2011).

Here, we have chosen a different route. The main reason for this is that the current presentation of the semantics brings out more explicitly how the notion of meaning is reshaped according to our needs. The algebraic perspective presented in section 3, then, allows us to motivate the clauses of $\mathsf{Inq}_{\mathrm{B}}$ in a solid way, relying only on the structure of our new space of meanings. Thus, unlike the support-based approach, which is only motivated *a posteriori*, this mode of presentation flows directly from the abstract motivations and the philosophical underpinnings of the system to its concrete implementation.

Moreover, given the intuitive interpretation of support in terms of insignificance (non-informativeness and non-inquisitiveness), the notion has a negative flavour to it, and its relation to the positive contribution of a sentence, as given in terms of its *potential*, is intuitively far from immediate. The current presentation focuses on this positive side, hopefully reflecting more directly how inquisitive semantics can be used in modeling conversation.

5 Inquisitive logic

In this section, we will be concerned with the logic that $\mathsf{Inq}_{\mathrm{B}}$ gives rise to. We will show that this logic is an intermediate logic, i.e., a logic in between classical and intuitionistic logic, which is not closed under uniform substitution. While relatively little is known about the general first-order system, with only some preliminary results in Ciardelli (2009), propositional inquisitive logic has been investigated in detail, and much is known about it, including a range of axiomatizations and its precise relation to classical and intuitionistic logic. We will present a number of results that hold for the general first-order case in section 5.1 and then zoom in on the propositional case in section 5.2.[5]

5.1 First-order inquisitive logic

Recall from section 4.1 that a sentence φ entails another sentence ψ in $\mathsf{Inq}_{\mathrm{B}}$ iff $[\varphi] \subseteq [\psi]$. We write $\varphi \models_{\mathsf{InqQL}} \psi$ in this case. We say that a formula φ is *valid* in inquisitive semantics iff it expresses a tautology, i.e., iff $[\varphi] = \wp(\omega)$.

[5]This section is based on Ciardelli (2009) and Ciardelli and Roelofsen (2011). For proofs and more comprehensive discussion of the logical issues discussed here, the reader is referred to these sources.

The set of first-order formulas that are valid in inquisitive semantics is called *inquisitive first-order logic* and is denoted by InqQL.

Definition 34 (Logic). $\mathsf{InqQL} := \{\varphi \mid [\varphi] = \wp(\omega)\}$.

As usual, the relation between entailment and validity is provided by the deduction theorem.

Fact 26 (Deduction theorem). For any two formulas φ and ψ:

$$\varphi \models_{\mathsf{InqQL}} \psi \iff \varphi \to \psi \in \mathsf{InqQL}$$

We will use IQL and CQL to denote intuitionistic and classical first-order logic respectively, and $\models_{\mathsf{IQL}}$ and $\models_{\mathsf{CQL}}$ to denote the corresponding entailment relations.

Now, if a formula φ expresses a tautology in inquisitive semantics, in particular it is not informative; therefore, by fact 13, it is a classical tautology. Formally, this simply amounts to the observation that if $[\varphi] = \wp(\omega)$, then $|\varphi| = \bigcup[\varphi] = \omega$. So, inquisitive logic in contained in classical first-order logic.

Fact 27. $\mathsf{InqQL} \subseteq \mathsf{CQL}$.

This inclusion is strict: for instance, in section 4 we saw that one of the most famous classical tautologies, $p \vee \neg p$, expresses a polar question in inquisitive semantics, not a tautology.

However, there is an important class of formulas on which inquisitive and classical logic coincide: assertions. This is to be expected, since assertions are those formulas whose meaning consist exclusively of informative content, just like in the classical case.

Fact 28 (Assertions behave classically). If φ and ψ are assertions,

$$\varphi \models_{\mathsf{InqQL}} \psi \iff \varphi \models_{\mathsf{CQL}} \psi$$

In particular, since any formula which does not contain disjunction or the existential quantifier is an assertion (corollary 1), inquisitive logic coincides with classical logic on the whole $\vee, \exists-$free fragment of the language.

There is also something interesting to say about the other most famous classical tautology, i.e., the law of double negation, $\neg\neg\varphi \to \varphi$.

Fact 29. $\neg\neg\varphi \to \varphi$ is in InqQL iff φ is an assertion.

This means that assertions are *precisely* the class of formulas on which inquisitive and classical entailment coincide.

Notice that the double negation law holds for atoms, which are assertions, but fails to hold in general. This simple fact displays a peculiar feature of InqQL: it is not closed under uniform substitution. That is, if $\varphi \in \mathsf{InqQL}$ and we replace the atoms in φ with arbitrary formulas, we have no guarantee that the resulting formula will be valid as well.

This is due to the fact that, in inquisitive semantics, atomic formulas do not express generic propositions. They are assertions, and thus have the special property of being equivalent with their own double negation. Indeed, one can prove that substituting atoms by assertions in a valid formula always results in a valid formula.

In section 3 we have seen that the algebra $(\Pi, \subseteq)$ of inquisitive meanings forms a complete Heyting algebra. Our semantics simply amounts to evaluating formulas in this particular structure, interpreting the logical constants as the corresponding algebraic operations. In this perspective, InqQL is the theory of a particular Heyting-valued model. Since intuitionistic first-order logic can be characterized as the set of those formulas valid in all Heyting-valued models (Troelstra and van Dalen, 1988), we have the following result.

Fact 30. $\mathsf{IQL} \subseteq \mathsf{InqQL}$.

This inclusion is also strict. For instance, we have seen that the double negation law holds for atoms in InqQL, whereas this is not the case in IQL.

Summing up the results in this section: InqQL is a logic lying in between intuitionistic and classical first-order logic, not closed under uniform substitution. A number of non intuitionistically valid principles are known to hold in InqQL (see Ciardelli, 2009). However, the axiomatic characterization of InqQL remains an open problem.

5.2 Propositional inquisitive logic

In this section, we will restrict our attention to inquisitive logic for a propositional language, which we denote as InqPL. Intuitionistic and classical propositional logic will be denoted by IPL and CPL respectively.

It is immediate that all the facts stated in the previous section for InqQL also hold for InqPL. In particular, InqPL is an intermediate logic that is not closed under uniform substitution. But what logic is it exactly? Can it be described by means of a natural axiomatization?

This time, we will be able to answer these questions in the positive. Let's start once again from classical logic. In that setting, all formulas are assertions. On the proof-theoretic side, this is witnessed by the fact that any formula is equivalent to a negation, for instance to its own double negation.

Fact 31. There exists a recursively defined map nt s.t. for any φ:

1. $\mathsf{nt}(\varphi)$ is a negation;
2. $\varphi \equiv_{\mathsf{CPL}} \mathsf{nt}(\varphi)$.

Once classical meanings are expressed by means of negations, any difference between classical and intuitionistic logic vanishes.

Fact 32. If φ and ψ are negations, then:

$$\varphi \models_{\mathsf{CPL}} \psi \iff \varphi \models_{\mathsf{IPL}} \psi$$

This means that, in a very precise sense, classical logic can be regarded as the negative fragment of intuitionistic logic, and the map nt—commonly known as *negative translation*—provides an embedding.

The situation is very similar for inquisitive logic. Now, formulas are no longer assertions in general. However, in the propositional setting, a formula is always equivalent to a *disjunction* of assertions, that is, to a disjunction of negations. This syntactic feature reflects the fact that inquisitive meanings are sets (disjunctions) of classical meanings (negations).

Fact 33. There exists a recursively defined map dnt s.t. for any φ:

1. $\mathsf{dnt}(\varphi)$ is a disjunction of negations;
2. $\varphi \equiv_{\mathsf{InqPL}} \mathsf{dnt}(\varphi)$.

Moreover, once an inquisitive meaning of a formula is expressed as a disjunction of negations, inquisitive and intuitionistic logic coincide.

Fact 34. If φ and ψ are disjunctions of negations, then

$$\varphi \models_{\mathsf{InqPL}} \psi \iff \varphi \models_{\mathsf{IPL}} \psi$$

This means that inquisitive logic can be regarded as the disjunctive-negative fragment of intuitionisitic logic, and that dnt—which we refer to as the *disjunctive-negative translation*—provides an embedding.

Thus, the algebra of intuitionistic meanings[6] provides a very rich environment, in which certain particular meanings, those associated to disjunctions of negations, correspond to the meanings of inquisitive semantics; among these, in turn, the meanings associated to negations represent the spectrum of classical meanings, that is, of meanings of assertions.

The disjunctive-negative translation dnt provides the key to understanding the logical properties of propositional inquisitive logic. As we have seen, it clarifies the way in which InqPL sits in between IPL and CPL. But it does more than that: it also paves the way for a completeness result.

For, suppose that L is an extension of intuitionistic logic which "justifies the disjunctive negative translation", in the sense that for any φ, L proves that φ and $\mathsf{dnt}(\varphi)$ are equivalent. We write $\vdash_L$ for provability in L.

Consider a valid inquisitive entailment, $\varphi \models_{\mathsf{InqPL}} \psi$. Then we also have that $\mathsf{dnt}(\varphi) \models_{\mathsf{InqPL}} \mathsf{dnt}(\psi)$ and therefore by proposition 34, $\mathsf{dnt}(\varphi) \models_{\mathsf{IPL}} \mathsf{dnt}(\psi)$. But then, since L extends IPL, $\mathsf{dnt}(\varphi) \vdash_L \mathsf{dnt}(\psi)$.

Since L justifies dnt, we also have $\varphi \vdash_L \mathsf{dnt}(\varphi)$ and $\mathsf{dnt}(\psi) \vdash_L \psi$, whence, putting everything together, $\varphi \vdash_L \psi$. Therefore, L is a complete derivation system for InqPL.

We are only left with the task of identifying what is needed, on top of intuitionistic logic, to justify dnt. Analyzing the inductive definition of the map dnt, we see that two extra ingredients suffice.

1. Atomic double negation axioms $\neg\neg p \to p$, needed for the translation of atoms;

2. For any number n, any instance of the following scheme, needed to justify the translation of implication:

   $$(\mathtt{ND}_n) \qquad (\neg\varphi \to \textstyle\bigvee_{1\leq i\leq n} \neg\psi_i) \to \textstyle\bigvee_{1\leq i\leq n}(\neg\varphi \to \neg\psi_i)$$

   The intermediate logic that is obtained by expanding IPL with the schemata $\mathtt{ND}_n$ for all natural numbers n is called ND. It has been investigated by Maksimova (1986).

[6] Here we use the word *meanings* for equivalence classes of formulas; for classical and inquisitive logic, for which we have a semantical notion of meaning, there is a bijective correspondence between the two notions.

These considerations yield the following sound and complete axiomatization of InqPL.

Theorem 1 (Completeness theorem)**.** ND augmented with double negation for atoms constitutes a sound and complete axiomatization of InqPL.

The infinite family of axioms $\{\mathtt{ND}_n \mid n \in \mathbb{N}\}$ may be substituted by a single, stronger axiom, known as the Kreisel-Putnam axiom.

$$(\mathtt{KP}) \qquad (\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$$

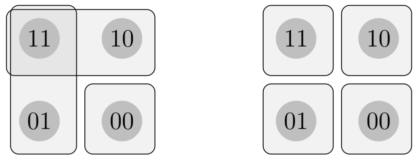
The Kreisel-Putnam logic KP is obtained by expanding IPL with the schema $\mathtt{KP}$. The logic KP is strictly stronger than Maksimova's logic ND. However, when augmented with atomic double negation axioms, both logics amount to the same thing, namely InqPL.

Theorem 2 (Completeness theorem)**.** KP augmented with double negation for atoms also constitutes a sound and complete axiomatization of InqPL.

We have thus reached what is perhaps the most elegant axiomatization of InqPL. However, it should be remarked that there is a whole range of intermediate logics Λ which, expanded with atomic double negation, yield InqPL. In Ciardelli (2009); Ciardelli and Roelofsen (2011), this range is precisely characterized as the set of logics included between ND and a logic called Medvedev's logic.

6 Relevance for natural language semantics

So far, we have motivated inquisitive semantics at a rather abstract level, specified a concrete system, Inq_B, and investigated the main features of this basic system. In this section, we discuss the relevance of inquisitive semantics, and in particular of Inq_B, for natural language semantics. We start with a very general discussion of the role that inquisitive semantics is intended to play in the semantic analysis of natural language. Then we discuss three different perspectives on inquisitiveness that are all in principle compatible with the basic philosophy behind the framework. Finally, we discuss the specific treatment of the logical connectives and the projection operators in Inq_B, and the possible significance thereof for natural language semantics, paying particular attention to disjunction and existential quantification as sources of inquisitiveness.



(a) $?\exists x.Px$ (b) $\forall x.?Px$

Figure 9: Two possible analysis of *wh*-interrogatives.

6.1 Inquistive semantics as a semantic framework

Inquisitive semantics is, first and foremost, intended to serve as a *framework* for natural language semantics. This means that it does not constitute a specific *theory* of any particular construction in any particular language. Rather, it is intended to provide the formal tools that are necessary to formulate, compare, and further develop such theories.

This is perhaps best illustrated by means of an example. Consider the *wh*-interrogative in (1).

(1) Who is coming to the party?

$\mathsf{Inq_B}$ provides a certain space of meanings, and associates these meanings in a particular way with formulas in a first-order language. But it leaves open how these meanings/formulas should be linked to sentences in natural language. For instance, the sentence in (1) may be associated with the formula $?\exists x.Px$ and the corresponding meaning, in line with Hamblin's (1973) and Karttunen's (1977) analysis of *wh*-interrogatives, illustrated in figure 9(a). But it may also be associated with the formula $\forall x.?Px$ and the corresponding meaning, in line with Groenendijk and Stokhof's (1984) partition theory of *wh*-interrogatives, illustrated in figure 9(b). Both type of theories may be formulated, compared, and possibly adapted or combined within the framework.

6.2 Three perspectives on inquisitiveness

The proposition $[\varphi]$ expressed by a sentence φ in $\mathsf{Inq_B}$ embodies the issue that is raised in uttering φ. In order to settle this issue, other participants must provide enough information to establish one of the states in $[\varphi]$. Moreover, we

have been assuming implicitly that, when raising an issue in a conversation, a speaker *requests* a response from other participants that settles the issue. However, the latter assumption is independent of the basic philosophy behind the framework, and is not reflected in any way by its formal implementation. Alternatively, we may just as well assume that a speaker, when raising an issue in a conversation, merely *invites* a response from other participants that settles the issue. Or we may adopt a more context-sensitive perspective, and assume that in uttering a sentence φ in a state s, a speaker requests a response that settles $[\varphi]$ just in case φ is purely inquisitive (not informative) in s. These three perspectives on inquisitiveness are listed below.

1. **The strong perspective.** In uttering a sentence φ in a state s, a speaker *requests* a response from other participants that provides enough information to establish one of the states in $[\varphi]_s$.

2. **The weak perspective.** In uttering a sentence φ in a state s, a speaker *invites* a response from other participants that provides enough information to establish one of the states in $[\varphi]_s$.

3. **The context-sensitive perspective.** In uttering a sentence φ in a state s, a speaker *invites* a response from other participants that provides enough information to establish one of the states in $[\varphi]_s$, and she *requests* such a response just in case φ is purely inquisitive (not informative) in s.

As mentioned above, we have assumed the strong perspective in these notes. But the other perspectives are equally viable, and have indeed been adopted elsewhere. For instance, the weak perspective can be found in Groenendijk (2009), and the context-sensitive perspective in AnderBois (2011).

This said, it is important when using inquisitive semantics as a framework for linguistic analysis to always explicitly choose a particular perspective on inquisitiveness. After all, the predictions of a theory formulated in the framework are partly determined by the perspective on inquisitiveness that is assumed. To illustrate this point, consider the following disjunctive declarative sentence.

(2) Alf or Bea will play the piano tonight.

Suppose we have a theory that associates this sentence with the formula $Pa \vee Pb$ and the corresponding meaning. This means that the sentence is predicted to be inquisitive. If we adopt the weak perspective on inquisitiveness, this prediction is reasonable. However, if we adopt the strong perspective on inquisitiveness, the prediction is wrong. After all, in uttering (2) a speaker does not request a response from other participants that establishes that Alf will play the piano or that Bea will play the piano. Under a strong perspective on inquisitiveness, (2) should not be associated with $Pa \vee Pb$ but rather with $!(Pa \vee Pb)$ and the corresponding meaning, which is not inquisitive.

Under the weak perspective, certain basic contrasts cannot be captured. Consider, for instance, the polar interrogative in (3), in comparison with the disjunctive declarative in (2).

(3) Will Alf play the piano tonight?

There is a clear intuition that (3) is inquisitive in a strong sense: in uttering this sentence, a speaker does not just invite, but really *requests* a response from other participants that establishes whether Alf will play the piano or not. In this sense there is a clear contrast between (2) and (3). Under the strong perspective, this contrast can be captured straightforwardly. Under the weak perspective, it cannot be captured because sentences, even if inquisitive, are never predicted to request an informative response.

6.3 Basic operations on meanings

We identified certain basic operations on meanings in $\mathsf{Inq}_{\mathrm{B}}$: the algebraic *meet*, *join*, and *(relative) pseudo-complement* operators, as well that the non-informative and non-inquisitive projection operators. We defined a semantics for the language of first-order logic, in which the basic logical connectives, as well as ! and ?, are associated with these basic operations on meanings: conjunction behaves as a *meet* operator, disjunction behaves as a *join* operator, ! behaves as a non-inquisitive projection operator, etcetera.

Of course, natural languages are much more intricate than the language of first-order logic. However, if we take sentences in natural language to have the type of meanings considered here, then it is natural to expect that it is also possible in these language to express the basic operations on such meanings. In other words, it is to be expected that natural languages will generally have certain words or constructions whose semantic function (possibly among others) is to produce, say, the non-informative projection of a proposition, or the meet of two propositions.

It seems plausible to assume that in the specific case of English, the meet of two propositions is constructed using the word *and*, the join is constructed using *or*, the relative pseudo-complement is constructed using *if. . . then*, and the pseudo-complement is constructed using *not*. This is not to say that this is the only semantic function that these words may have. But the expectation that the language makes it possible to express the basic algebraic operations on meanings seems to be borne out in the case of English, and many other languages alike.

As for the projection operators, it seems plausible to hypothesize that these are expressed in English and many other languages by declarative and interrogative *complementizers*. More specifically, it seems plausible to treat the declarative complementizer in English as !, the *wh*-interrogative complementizer as ?, and the polar interrogative complementizer as ?!. A detailed examination of this analysis is beyond the scope of these notes. Importantly, however, note that the framework also allows us to formulate alternative analyses. As emphasized above, the framework as such does not make any direct predictions about the semantic behavior of any specific constructions in any specific language. It mainly offers the logical tools that are necessary to formulate such analyses, and gives rise to the expectation that, in general, natural languages will have ways to express the basic algebraic operations and the basic projection operations on meanings.

6.4 Inquisitive disjunction and indefinites

Among the basic operations on meanings that we have considered, the *join* operator is the essential source of inquisitiveness: without applying this operator, it is impossible to produce inquisitive meanings from non-inquisitive ones. In $\mathsf{Inq}_{\mathrm{B}}$, disjunction, the existential quantifier, and the non-informative projection operator all behave as join operators: $[\varphi \vee \psi]$ is the join of $[\varphi]$ and $[\psi]$, $[\exists x.\varphi(x)]$ is the join of $\{[\varphi(d)] \mid d \in D\}$, and $[?\varphi]$ is the join of $[\varphi]$ and $[\varphi]^*$. Thus, there is a close connection between the non-informative projection operator, ?, which is naturally associated with interrogative complementizers in natural languages, and disjunction / existential quantification. All these constructions are sources of inquisitiveness. This fact may provide the basis for an explanation of the well-known observation that in many natural languages, interrogative pronouns/complementizers are homophonous with words for disjunction and/or indefinites (e.g., Japanese *ka*) (see Jayaseelan, 2001, 2008; Bhat, 2005; Haida, 2007; AnderBois, 2011, among others).

It is also interesting to note that there is a close connection between the treatment of disjunction and existential quantification in $\mathsf{Inq}_{\mathrm{B}}$, and their treatment in *alternative semantics* (Kratzer and Shimoyama, 2002; Simons, 2005a,b; Alonso-Ovalle, 2006, 2008, 2009; Aloni, 2007a,b; Menéndez-Benito, 2005, 2010, among others). In both cases, disjunction and existentials are taken to introduce sets of alternatives. In the case of alternative semantics, this treatment is motivated by a number of empirical phenomena, including free choice inferences, exclusivity implicatures, and conditionals with disjunctive antecedents. The analysis of disjunction and existentials as introducing sets of alternatives has made it possible to develop new accounts of these phenomena which improve considerably on previous accounts. However, alternative semantics does not provide any motivation for the alternative treatment of disjunction and indefinites *independently* of the linguistic phenomena at hand. Moreover, the treatment of disjunction in alternative semantics has been presented as a real *alternative* for the classical treatment of disjunction as a *join* operator. Thus, it appears that adopting the alternative treatment of disjunction forces one to give up the classical account.

The algebraically motivated inquisitive semantics presented here sheds new light on these two issues. First, it shows that, once inquisitive content is taken into consideration besides informative content, general algebraic considerations lead essentially to the treatment of disjunction that was proposed in alternative semantics, thus providing exactly the independent motivation that has so far been missing. Moreover, it shows that the 'alternative' treatment of disjunction is actually a natural generalization of the classical treatment: disjunction is still taken to behave semantically as a *join* operator, only now the meanings that this join operator applies to are more fine-grained in order to capture both informative and inquisitive content. Thus, we can have our cake and eat it: we can adopt a treatment of disjunction as introducing sets of alternatives, and still characterize it as a *join* operator.

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