Social Choice Theory for Logicians
NASSLLI 2012

Lecture 1 Handout

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June 17, 2012

1 Arrow's Theorem

Notation

- X is a (finite or infinite) set of **alternatives** (or **candidates**.
- $N = \{1, \ldots, n\}$ is a set of **voters**
- **Preferences**: $\mathcal{P} = \{R \mid R \subseteq W \times W$ where R is reflexive, transitive and connected$\}$
- Given $R \in \mathcal{P}$, let P be the **strict preference generated by** R: xPy iff xRy and not yRx (we write P_R if necessary)
- A profile is a tuple $(R_1, \ldots, R_n) \in \mathcal{P}^n$
- **Social Welfare Function**: $F : D \to \mathcal{P}$ where $D \subseteq \mathcal{P}^n$ is the domain.

Axioms

- **Universal Domain** (UD): The domain of F is $\mathcal{P}^n$ (i.e., $D = \mathcal{P}^n$)
- **Independence of Irrelevant Alternatives** (IIA): F satisfies IIA provide for all profiles $\vec{R}, \vec{R}' \in D$, if xR_iy iff xR'_iy for all $i \in N$, then $xF(\vec{R})y$ iff $xF(\vec{R}')y$
- **(weak) Pareto** (P): For all profiles $\vec{R} \in D$, if xP_iy for all $i \in N$ then $xP_{F(\vec{R})}y$

- Agent i is a **dictator** for F provided for every preference profile and every pair $x, y \in X$, xP_iy implies $xF(\vec{R})y$.

Arrows (Im)possibility theorem: Suppose that $\|X\| \geq 3$ and F satisfies UD, IIA and P. Then there is some $i \in N$ that is a dictator for F.

Key Lemmas

First, some key definitions. To simplify the notation, for a $\vec{R} \in D$, we write S for the social ordering given by F, i.e., $F(\vec{R}) = Q$.

For a set of voters $S \subseteq \{1, \ldots, n\}$, we say

- S is **decisive for** x **over** y if for some preference profile $\vec{R}$ we have xP_ix for all $i \in S$, yP_ix for all $i \notin S$ and xP_Qy.
- S is **strictly decisive for** x **over** y if for every preference profile $\vec{R}$ satisfying xP_iy for all $i \in S$, we have xP_Qy
- S is **decisive** if it is strictly decisive for every pair of distinct alternatives.

Lemma 1 Suppose that for some x and y, S is decisive for x over y, then S is decisive.

Lemma 2 If S and T are decisive then so is $S \cap T$

Lemma 3 If S is not decisive, then $S^C = N - S$ is decisive.

Arrow's Theorem: There is a singleton decisive set.

K. Arrow, *Social Choice and Individual Values*, Yale University Press, 1951 (2nd edition, 1963).

2 Sen's Theorem

Notation

- *Linear Preferences*: $\mathcal{L} = \{> \mid <\subseteq X \times X \text{ is a linear order}\}$
- **Social choice function**: $C : \mathcal{L}^n \to X$

Axioms

- **(weak) Unanimity**: C satisfies weak unanimity provided if for every preference profile $> \in \mathcal{L}^n$, if there is a pair of alternatives x and y such that $x >_i y$ for all $i \in N$, then $C(>) \neq y$.

- **Liberalism** C satisfies liberalism provided if for every individual i, there exists two distinct alternatives $x, y \in X$ such that i is two-way decisive on x and y: If $x >_i y$, then $C(>) \neq y$; and if $y >_i x$, then $C(>) \neq x$

Sen's Impossibility of the Paretian Liberal: No social choice function satisfies both liberalism and the weakly unanimity conditions.

A. K. Sen. The impossibility of a Paretian liberal. *The Journal of Political Economics*, 78 (1):152 - 157, 1970.

3 Muller-Satterthwaite Theorem

Axioms

- **Monotonicity**: C is monotonic provided if for every preference profile $> \in \mathcal{L}^n$ such that $C(>) = x$, if $>'$ is another profile such that $x >'_i y$ whenever $x >_i y$ for every agent i and alternative y, then $C(>') = x$.

- **Dictator**: A voter i is a dictator in a social choice function C if C always selects is top choice: for every preference profile $>$, $C(>) = a$ iff for all $y \in X$ different from x, $x >_i y$.

Muller-Satterthwaite Theorem: If $|X| \geq 3$, then any social choice function that is weakly unanimous and monotonic is also dictatorial.

Proof

- A set of voters S is **winning** if, for any profile $> \in \mathcal{L}^n$ in which every $i \in S$ ranks some alternative x on top of her preference, $C(>) = x$.

- A set of voters S is **blocking** if there exists some profile $> \in \mathcal{L}^n$ such that $C(>) = x$ and x is ranked on the top by all $i \in S$ and ranked at the bottom for all $i \in N - S$.

- **Theorem (Tang and Sandholm)**. If $|X| \geq 3$, then for any social choice function satisfying weak unanimity and strong monotonicity, a coalition is winning iff it is blocking.

E. Muller and M. A. Satterthwaite. The equivalence of strong positive association and strategy-proofness. *Journal of Economic Theory*, 14(2):412-418, 1977.

P. Tang and T. Sandholm, Coalitional Structure of the Muller-Satterthwaite Theorem. In *Proceedings of the Workshop on Cooperative Games in Multiagent Systems* (CoopMAS) at AAMAS, 2012.